Sequences and Series Lecture Notes

Introduction

Although much of the mathematics we've done in this course deals with algebra and graphing, many mathematicians would say that in general mathematics deals with patterns, whether they're visual patterns or numerical patterns. For example, exponential growth is a growth pattern that is shared by populations, bank accounts etc.

Sequences and Series deal with numerical patterns. We'll start with what a sequence is.

Sequences

We've all come across the plain English definition of a sequence. For example, when you find the DNA sequence of a mouse, it's an ordered list of DNA proteins. Similarly, in mathematics, a sequence is an ordered list of numbers following some pattern, for example,

1, 4, 7, 10, ...

In this sequence, the pattern is that I started with a 1, and add 3 to get the next term (the name for the elements of the sequence), and so on. Once you know the pattern you can figure out extra terms (or even go on for ever figuring out terms), for example the next few terms are

1, 4, 7, 10, 13, 16, 19, ...

Notation and Formulas for Sequences:

Now that you've got the intuitive idea of what a sequence is, let's look at some notation and terminology.

The ingredients of a sequence are called terms (this is to distinguish them from the elements of a set - a set has no particular order but a sequence does). If we need to use variables for the names of the terms, we use subscripts to say which term we're talking about. For example, if I use the letter $a$ for the name of the sequence above, we have that

$a_1 = 1, a_2 = 4, a_3 = 7, ...$

Now I can use this notation to give the sequence as a formula rather than a list. For example, the formula in general for the above sequence is

$a_n = 1 + (n - 1) \cdot 3$

Where $n$ is any integer, 1, 2, 3, etc. The advantage of giving the formula rather than a list is that you can get any term you like without getting all the rest. For example, the 20th term of our sequence above is

$a_{20} = 1 + (19) \cdot 3 = 1 + 57 = 58$

This is more without the formula, since I would have had to list lots of terms.
Now that you understand the basic labeling for sequences we can look at the notation for specifying a sequence using a formula rather than a list of terms. It’s easiest for me to show with an example.

Example: Write out the first 4 terms of the sequence given by

\[ \{u_n\} = n^2 + 2 \]

Solution: Instead of a list of the first few terms, we’ve been given the formula getting any term we like. The way you read this notation is this. The left hand side uses set notation, since a sequence is just a special type of set where the elements are ordered. It gives the name of the series as \( u \), so the terms are called \( u_1, u_2, u_3 \) and so on. The rule for getting the elements (or terms, as we call them) is given on the right side.

So the above formula says that to get any particular term (the “nth” term) we square \( n \) and add 2. So the first few terms are (just setting \( n=1,2,3,4 \) are:

\[
\begin{align*}
  u_1 &= (1)^2 + 2 = 3, \\
  u_2 &= (2)^2 + 2 = 6, \\
  u_3 &= (3)^2 + 2 = 11, \\
  u_4 &= (4)^2 + 2 = 18
\end{align*}
\]

So the first 4 terms of the sequence are 3, 6, 11, 18.

If you’ve spotted some similarity between function notation and sequence notation (for example, the name is on the left of the =, and the rule is on the right), it’s not a coincidence. In fact, in formal terms a sequence is just a function where the domain is the integers. In fact, we could just define our sequence \( u \) using function notation

\[ u(n) = n^2 + 2, \quad n = 1, 2, 3, 4 \ldots \]

-but if we do this we have to keep reminding the reader than the inputs are integers. So it’s nice to have a special notation just for sequences to avoid confusion.

Sometimes the pattern the sequence follows can be very complicated, and figuring out the formula is tough. Fortunately, in this introduction, we only look at very simple patterns, namely arithmetic sequences and geometric sequences.
Arithmetic Sequences

In an arithmetic sequence I generate the sequence by adding (or subtracting) a constant from a particular term to get the next term. So the difference between successive terms is constant for an arithmetic sequence. Look again at the sequence:

\[ 1, 4, 7, 11, \ldots \]

And you see the difference between each pair of terms is 3.

Another example. Here I’ve started at 10, and I’m subtracting 2 from each term to get the next one.

\[ 10, 8, 6, 4, \ldots \]

Formulas for Arithmetic Sequences: If you’re given an arithmetic sequence as a list and asked to figure out the formula, the key is to determine the first term (we’ll call this \( a \)) and the difference between successive terms (we’ll call this \( d \)). Then the formula for the \( n \)th term, where \( n \) is any integer, is

\[ a_n = a + (n - 1)d \]

Example. Find an expression for the \( n \)th term of the sequence \(-2, 4, 10, 16, \ldots \). And use this formula to find the 15th term of the sequence.

Solution. You can easily see that the difference between successive terms is 6, and we start at -2. Note that the difference between all the successive terms must be the same or it’s not an arithmetic series. So here \( a = -2 \) and \( d = 6 \). So the general formula is

\[ a_n = a + (n - 1)d \]

\[ a_n = -2 + (n - 1)(6) \]

Now the 15th term is

\[ a_{15} = -2 + (14)(6) = 82 \]

Now we look at geometric sequences.
Geometric Sequences

In a geometric sequence the ratio between successive terms is constant, - to get the next term I multiply the current term by a constant.

For example: 3, 6, 12, 24, …

The next term in this sequence is of course 48, since each term is twice the last one.

Here’s another example that is geometric, see if you can work out what the ratio is:

3, -1, 1/3, -1/9, …

To get the next term I multiply the current term by -1/3. So the constant ratio doesn’t have to be greater than one, it can be less, like in the example I just did.

Formulas for Geometric Sequences: If you’re given a geometric sequence as a list and asked to figure out the formula, the key is to determine the first term (we’ll call this \( a \)) and the ratio between successive terms (we’ll call this \( r \)). Then the formula for the nth term, where \( n \) is any integer, is

\[
a_n = ar^{n-1}
\]

Note that the exponent only applies to the ratio bit. Also note the exponent is \( n-1 \), not \( n \).

Example. Figure out the 8th term of

3, -1, 1/3, -1/9, …

Solution. The ratio between each pair of terms is \(-\frac{1}{3}\), since you multiply the current term by \(-\frac{1}{3}\) to get the next one (for example, to get \( a_2 \) we multiply \( a_1 \) which is 3 by \(-\frac{1}{3}\), giving -1. We multiply -1 by \(-\frac{1}{3}\) to get \( a_3 = \left(-\frac{1}{3}\right)(-1) = \frac{1}{3} \).

And so on. So \( a = 3 \) and \( r = -\frac{1}{3} \). The nth term is \( a_n = ar^{n-1} = 3 \left(-\frac{1}{3}\right)^{n-1} \). So the 8th term is \( a_8 = 3 \left(-\frac{1}{3}\right)^7 \) which is \( a_8 = -\frac{1}{729} \approx 0.00137 \)

We’ll do series next, but before we do, it’s important to stress that arithmetic and geometric sequences are special types of sequence. Not every sequence is one of these two types, and most are neither arithmetic nor geometric. For example, the sequence 1, 4, 9, 16, has an obvious pattern
\{u_n\} = n^2$ but the series is neither arithmetic (the differences aren’t constant) or geometric (since the ratios aren’t constant).

**Series - The Idea and Notation**

A series is built from a sequence, but differs from it in that the terms are added together. For example

$$1, 4, 7, 11, \ldots$$

Is a sequence, but

$$1 + 4 + 7 + 11 + \ldots$$

Is a series.

A series can be finite (for example, it might only have 25 terms) or infinite, and the notation needs to allow for both.

The easiest way to get used to series notation is with an example.

**Example:** Find the sum of the series

$$\sum_{i=1}^{5} 3i + 4$$

**Solution:** The symbol $\sum$ is called a ‘sigma’ (it’s a Greek S, for ‘SUM’) and this notation is called ‘sigma notation’. It’s easy enough to read. Below the sigma the variable name we are going to use for each term is given\(^1\), and the value to start it at (generally 1, but some start at 0 in calc 3 for technical reasons I won’t bore you with). Above the sigma is when to stop (or you can put in an infinity symbol to go on for ever). Finally, to the right of the sigma is the rule for calculating each term of the series.

The series above has the rule:

“for each number $i$ from 1 to 5, plug them into the rule $3i + 4$ and add up the total”.

So here

$$\sum_{i=1}^{5} 3i + 4 = (3(1) + 4) + (3(2) + 4) + (3(3) + 4) + (3(4) + 4) + (3(5) + 4)$$

$$= 7 + 10 + 13 + 16 + 19$$

$$= 65$$

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\(^1\) We usually use lower case ‘$i$’ rather than ‘$n$’ for series formulas, because sometimes we will want to reserve $n$ for the number of terms, rather than the current term.
If you want to give the series a name you can do this by incorporating our sequence notation (we know a series is just a summed up sequence). So we could have phrased this question as

Find $\sum_{i=1}^{5} u_i$ where $u_i = 3i + 4$

**Convergence and Divergence**

If the series goes on forever (and in real world applications many do) adding the terms might seem a bit pointless, since they seem to just add up to infinity. But some series, even though they go on forever, have a finite sum. For example, consider the series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

You can check by adding up this series for a couple more terms that it never climbs above 2, and in fact it approaches 2 the more terms you add up. This is called a convergent series, and this series converges to 2. You’ll meet more convergent series in calculus.

The sequences we have met generally go on for ever (at least in theory, in practise we only work with the first few terms most of the time), so a sequence can converge or diverge too. For example, the sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$

Converges to 0, but the sequence

$$2, 4, 6, 8, \ldots$$

Diverges, since it’s terms don’t tend to a constant value as $n$ gets large.

**Arithmetic Series**

If we have an arithmetic sequence, adding up the terms gives us an arithmetic series. Once we realize it’s arithmetic, and we get the rule for each term in the form

$$a_n = a + (n - 1)d$$

Then there’s an easy formula for adding up the first $n$ terms of an arithmetic series. It’s

$$S_n = n \left( \frac{a + a_n}{2} \right)$$

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2 Figuring out whether a series is convergent or not is harder than it looks. It’s not enough for the terms to be getting smaller as $n$ gets large. For example, the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots + \frac{1}{n} + \ldots$$

looks like it might converge (i.e. have a finite sum), but it does not. A series that doesn’t converge is said to diverge. Wait till Calc 3 to look at rules for testing series.
Where $S_n$ means the ‘sum of the first n terms’ (we could also use sigma notation, but this is briefer sometimes).

You’ll note that $a$ is the first term of the series, and $a_n$ is the last term. So our formula is pretty easy, it’s just taking the average of the first and last terms, and multiplying by the number of terms.

**Example:** Find the sum of the series

$$\sum_{i=1}^{25} 3i + 4$$

Now this is the same rule as we added up in the previous example, but having 25 terms makes it very hard to do by hand. If it’s arithmetic, however we can use the formula to avoid having to write out the sequence in full.

Now the first few terms of the series are 7+10+13+…. So clearly it’s arithmetic, with $a = 7$ and $d = 3$ (d, remember, is the difference between successive pairs of terms).

Now the formula is

$$a_i = a + (n-1)d$$

$$a_i = 7 + (i-1)(3) = 7 + 3i - 3 = 3i + 4$$

(which is the formula we started with!)

So now that we’ve shown it’s arithmetic, all we need to do to find the sum of the first 25 terms is use the formula

$$S_n = n \left( \frac{a + a_n}{2} \right)$$

$$S_{25} = 25 \left( \frac{a + a_{25}}{2} \right)$$

Now the first term is $a = 7$, and the 25th term is $a_{25} = 3(25) + 4 = 79$ so our formula is

$$S_{25} = 25 \left( \frac{7 + 79}{2} \right) = 1075$$

So the sum of the first 25 terms is 1075.
Geometric Series

Similarly, if we have an geometric sequence, adding up the terms gives us an geometric series. Once we realize it’s geometric, there’s a formula for the total sum.

Recall that a geometric sequence is for example: 3, 6, 12, 24, …

where the ratio between successive terms is constant. The ratio was labeled \( r \) and as before, the first term is labeled \( a \). Then the \( i \)th term of the sequence is from before

\[ a_i = ar^{i-1} \]

And the formula for the sum of the first \( n \) terms of a geometric series is

\[ S_n = \frac{a(1 - r^n)}{1 - r} \]

Furthermore, if we want to add all the terms of an infinite geometric series it’s given by

\[ S_\infty = \frac{a}{1 - r} \quad |r| < 1 \quad (\ast) \]

So as long as the ratio has an absolute value less than 1 the geometric series converges. (Otherwise it diverges).

Example: For the series

\[ 2 + \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + ... \]

(a) Find the sum of the first 10 terms.

(b) If it converges, find \( S_\infty \).

(a): The series is geometric since the common ratio is \( \frac{1}{4} \) - each term gets multiplied by \( \frac{1}{4} \) to get the next term. So \( a = 2 \) and \( r = \frac{1}{4} \). Let’s plug \( n = 10 \) into the above formula:

\[ S_{10} = \frac{2 \left(1 - \left(\frac{1}{4}\right)^{10}\right)}{1 - \frac{1}{4}} = 2.666664124 \]

(c) The sum to infinity is \( S_\infty = \frac{2}{1 - \frac{1}{4}} = \frac{8}{3} \).

That’s all for now. There’s much more on sequences and series that you can do in Calc 3 and beyond.